Equation (29) gives[12,13]

$$\Gamma(\omega \to \pi^0+\gamma)=1.2 \text{ MeV},$$

in agreement with experiment. The other transition moments, namely, $\mu_T(K^{+*} \to K^+ + \gamma)$, $\mu_T(K^{*0} \to K^0+\gamma)$ are related to (28) or (29) simply by SU_3 and so are $\mu_T(\rho^0 \to \eta+\gamma)$, $\mu_T(\omega \to \eta+\gamma)$, $\mu_T(\phi \to \eta+\gamma)$ if we neglect X^0, η^0 mixing.

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Theory of Reggeized Bootstraps*

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Using a representation for the partial-wave S matrix developed recently by the authors, a representation for the total scattering amplitude is constructed with Regge trajectories as the sole input. The resulting representation has the following properties regardless of the number of trajectories included: (a) It has the correct analytic structure; namely, the double spectral function is nonvanishing in a region bounded by the curved boundaries implied by the Mandelstam representation. (b) It is unitary. (c) It gives the correct threshold behavior for both the real and the imaginary parts of the total amplitude. (d) It converges rapidly. A potential theory example is worked out explicitly and with *one* trajectory input, numerically an excellent agreement with the exact result for the total amplitude is obtained. The representation is then generalized to the relativistic case. Invoking crossing, a set of relatively simple bootstrap equations is obtained for a self-consistent determination of the Regge trajectories. Also, an alternative procedure is proposed which imposes approximate crossing symmetry in the unphysical region to determine the Regge parameters. The latter method is operationally more attractive since it obviates solving any dispersion integrals, and also has the added advantage of avoiding inelasticity. Finally, in this method, no statements (or approximations) regarding the $I=2$ channel need be made, since the latter can be eliminated from the appropriate crossing relation.

I. INTRODUCTION

THE bootstrap hypothesis,[1] that all strongly interacting particles are composite systems of each other, though intuitively simple, is not easily put into a rigorous mathematical form[2] which can then be approximated in a consistent manner and thus meaningfully compared to experiment. The most encouraging statements still remain the signs and relative magnitudes of forces as determined from the crossing matrices. Quantitative methods suffer from various diseases.

By far the largest class of such calculations treat the exchanged particles (forces) as elementary and compute dynamic particles as output, requiring these to have the masses and couplings of the elementary input particles. Such a treatment violates the basic concept of the absence of elementary particles and crossing symmetry. Furthermore it also leads to the introduction of parameters, subtractions and cutoffs, to which the solutions are not insensitive. Whether such calculations can be considered an approximation to the bootstrap hypothesis is not clear, since they are not based on a consistent approximation scheme of a rigorously formulated theory.

In order to free the calculations of elementary particles, it is now plausible to exploit the fundamental connection between composite particles and Regge trajectories and attempt to bootstrap entire trajectories rather than just the position and slope at a single point. If this is done by solving the N/D equations, in a way similar to ordinary bootstrap calculations, then the problem is how to construct a "potential" from the exchanged trajectories. Alternatively, we may try to construct an amplitude which is both unitary and crossing symmetric in terms of the Regge trajectories and use these conditions to determine the parameters of the trajectories.

The most serious attempt so far to develop a consistent approximation scheme is the "strip approxi-

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[1] G. F. Chew and S. C. Frautschi, Phys. Rev. Letters **7**, 394 (1961).

[2] A most complete bibliography of the literature on bootstraps may be found in "Lectures on Bootstraps" by F. Zachariasen, given at the Pacific International Summer School in Physics, 1965, Honolulu, Hawaii (unpublished).

mation."[3] Here, the contribution from each pole arises from a distinct strip in the Mandelstam representation. The introduction of a definite, though somewhat arbitrary strip width s_0-s_1, not only brings in a parameter, but also gives straight boundaries to the double spectral function $\rho(s,t)$, instead of the correct curved ones. The resulting artificial singularities can be removed, but the value of the parameter s_1 is not unimportant in preliminary calculations.[4] The hope here is that it will become so when more channels are included. A modification of Kretzschmar, which also uses a coupling between trajectories and residues suggested elasehwere,[5] differs from the usual strip approximation by defining the strip width as the energy at which the Regge trajectory recedes into the left-half l-plane. The double spectral function of Kretzschmar also has the correct curved boundary. As was pointed out in a potential theory example, however,[6] even this rather large strip width is not large enough to serve as a reasonable dividing energy between "resonant" and "potential" scattering.[7]

In any approach to the Reggeized bootstrap problem, one crucial demand is to have a representation of the amplitude in terms of Regge poles, which not only converges, but converges as fast as possible. With present computational facilities, including trajectories other than those which reach or come near the right-hand l plane, in an actual calculation, is manifestly an impossible undertaking.

The present approach is based on a sequence of approximations which have been developed by considering Regge trajectories in potential theory.[6,8] The representation we will use[6] is a modification of one due to Cheng.[9] The modification, based on the asymptotic properties of all trajectories in the energy plane, brings about extremely rapid convergence in the potential theory examples studied. The residues $\beta(s)$ are simple functions of the trajectories $\alpha(s)$.

In Sec. II, we construct the total scattering amplitude within the framework of two-body unitarity and the nonrelativistic case. In Secs. III and IV we generalize to the relativistic case and suggest two alternatives to exploit the features of the representation. The first is to construct a "potential" to be used in the conventional N/D equations. The second is an alternative scheme, operationally much simpler, which determines the Regge parameters by imposing crossing symmetry at isolated points in the unphysical region.

In Sec. V, some numerical results are presented for a potential theory example. The comparison of the representation to the exact result reveals that, at least in the case tested, the leading trajectory represents the amplitude extremely well, but the influence of the trajectory persists to energies much higher than those for which it retreats to the left-hand l plane.

II. NONRELATIVISTIC TOTAL AMPLITUDE VIA THE MODIFIED CHENG REPRESENTATION

We would like to summarize first, some of the features of the modified Cheng representation,[6] which are pertinent to the present discussion. In this representation, the partial-wave S matrix in terms of its trajectories $\alpha_n(s)$ is given by

$$S(l,s)=\{\exp[(ig^2/\sqrt{s})Q_l(\cosh\xi)]\}\prod_n S_n(l,s)\,, \quad (1)$$

where

$$S_n(l,s)=\exp\left\{\int_{\alpha_n(s)}^{\alpha_n^*(s)}\frac{\exp[(l'-l)\bar{\xi}]}{l'-l}dl' -\frac{ig^2}{\sqrt{s}}\frac{\exp[-(l+n)\bar{\xi}]}{l+n}P_{n-1}(\cosh\xi)\right\}, \quad (2)$$

$$\cosh\xi(s)=1+\mu^2/2s\,,$$

and

$$\cosh\bar{\xi}(s)=1+2\mu^2/s\,,$$

with g^2 the strength of the Yukawa coupling and μ is its range. In the n-trajectory approximation when the product is limited only to n terms, the representation has several remarkable properties:

(a) $S(l,s)$ is unitary, no matter how many terms are kept in the infinite product.

(b) If the input trajectory $\alpha_n(s)$ is assumed to have the correct threshold behavior, the real and imaginary parts of the scattering amplitude also have the correct threshold behavior. The double spectral function in the Mandelstam representation would thus have the correct boundary. Other schemes, such as the Khuri representation, do not share this property.

(c) For a single Yukawa potential, the total amplitude given by (1) has the correct analytic behavior in the $\cos\theta$ plane; namely, it has a pole at $\cos\theta=1+\mu^2/2s$ from the Q function and a cut starting at $\cos\theta=1+2\mu^2/s$ from $\bar{\xi}(s)$. In Appendix C it will be shown that for a distribution of Yukawas with maximum range $1/\mu$, the generalization of (1) correctly changes the Born pole into a cut from $\cos\theta=1+\mu^2/2s$ to $\cos\theta=1+2\mu^2/s$.

As discussed in the earlier paper,[6] the representation was constructed so that the contribution from each Regge pole reflects the analytic properties of the total amplitude, so that it is not necessary to depend on the infinite collection of poles to produce the correct cut,

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[3] G. F. Chew, Phys. Rev. **129**, 2363 (1963) and G. F. Chew and C. E. Jones, Phys. Rev. **135**, B208 (1964).

[4] D. C. Teplitz and V. L. Teplitz, Phys. Rev. **137**, B142 (1965).

[5] M. Kretzschmar, Nuovo Cimento **39**, 835 (1965).

[6] W. J. Abbe, P. Kaus, P. Nath, and Y. N. Srivastava, Phys. Rev. **140**, B1595 (1965).

[7] This may, however, be due to the particularly nonresonant nature of the Yukawa potential, which makes the resulting s_1 rather low.

[8] S. C. Frautschi, P. Kaus, and F. Zachariasen, Phys. Rev. **133**, B1607 (1964); D. Hankins, P. Kaus, and C. J. Pearson, *ibid.* **137**, B1034 (1965).

[9] H. Cheng, Harvard University (to be published).

unitarity and symptotic behavior. Then, as expected, a rapid convergence to the exact result in terms of trajectories close to the physical right-hand l plane was verified by explicit numerical computation. One point which was numerically observed needs reassertion; it was found that at the energy for which the top trajectory recedes to the left of $\mathrm{Re}l=-0.5$, the S matrix is nowhere close to the Born limit, which would set $\mathrm{Re}S=1$, as can be seen from Fig. 1 of Ref. 6. Hence, if one wants to keep only the low-s part of the trajectory (strip approximation), the large-s part of the amplitude would be better represented by the relativistic analog of (1), with $n=0$, than by the nonunitary Born term.

The above representation then forms the starting point for obtaining a continuation of the total amplitude via the Watson-Sommerfeld transformation. We define the total amplitude $f(s,z)$ through

$$f(s,z)=\sum_{l=0}^{\infty}(2l+1)A_l(s)P_l(z)\,, \tag{3}$$

where $z=\cos\theta$, $0\leqslant\theta\leqslant\pi$, and

$$A_l(s)=\frac{S_l(s)-1}{2i\sqrt{s}}\,. \tag{4}$$

As usual, we obtain the continuation via the Watson-Sommerfeld transformation for all z:

$$f(s,z)=\frac{1}{\sqrt{s}}\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty}dl(2l+1)\left[\frac{S(l,s)-1}{4}\right]\frac{P_l(-z)}{\sin\pi l}$$
$$-\pi\sum_{\mathrm{Re}\alpha_n>-\frac{1}{2}}(2\alpha_n+1)\beta_n\frac{P_{\alpha_n}(-z)}{\sin\pi\alpha_n}\,, \tag{5}$$

where the $\beta_n(s)$ are the residues of $A(l,s)$ at the Regge poles $\alpha_n(s)$. In order to verify the convergence of the background integral,[10] we need only state that for interactions for which (1) is valid we have

$$A(-\tfrac{1}{2}+ip,\,s)=O\left(\frac{e^{-(-\frac{1}{2}+ip)\xi}}{\sqrt{p}}\right),\quad |p|\rightarrow\infty \tag{6}$$

and

$$\frac{P_{-\frac{1}{2}+ip}(-z)}{\sin\pi(-\frac{1}{2}+ip)}=O\left(\frac{e^{-|p|(\pi-|\eta|)}}{\sqrt{p}}\right),\quad |p|\rightarrow\infty \tag{7}$$

where $-\pi\leq\eta=\mathrm{Re}\cos^{-1}(-z)\leqslant\pi$. Hence, the integrand in (5) converges for $s\geqslant0$ and z not on the real axis between $z=1$ and $z=\infty$. The value of $f(s,z)$ on the cut must be obtained in a limiting sense, as one approaches the axis from above or below.

Now, if the input partial-wave S matrix has the correct analytic structure in the l plane, as it does, then the total amplitude will have the correct analytic structure in the s and t planes; in particular, it will have a pole at $z=1+\mu^2/2s$ and a cut beginning at $z=1+2\mu^2/s$ in the z plane, for a single Yukawa potential. This will be shown in Appendix A. As is well known, this statement is only true if both the Regge terms and the background integral in (5) are retained. The background integral cannot be discarded without impeding the correct analytic structure in the z plane. Therefore, we have a faithful representation, at least in the sense of potential theory, as the numerical results of this paper and of a previous paper[6] indicate; it is valid for $s\geqslant0$ and all z no matter how many trajectories are retained. Furthermore, the double spectral function $\rho(s,t)$, consistent with the above amplitude $f(s,z)$ will be nonvanishing in a region of the s-t plane bounded by the correct curve as required by the Mandelstam representation; this is shown in Appendix A.

Our procedure is now straightforward. We insert the input trajectories in (1) and compute the partial-wave amplitude; these amplitudes are then used in (5) to compute the total amplitude $f(s,z)$ valid for s above threshold and all z. In the next section we connect it to its "unphysical" region via relativistic crossing and thus close the loop.

III. RELATIVISTIC TOTAL AMPLITUDE

We shall consider, as a specific example, the problem of elastic pion-pion scattering. Let us define the usual s, t, u variables for this problem;

$$s=4(\nu+\mu_0^2)\,,$$
$$t=-2\nu(1-z)\,,$$
$$u=-2\nu(1+z)\,,$$

where $\nu=q_s^2$, the square of the center-of-mass momentum in the s channel and μ_0 is the pion mass. The isotopic spin index will be displayed explicitly, so that for $s\geqslant4\mu_0^2$ the total amplitude in the s channel may be written

$$A_s^I(s,t,u)=\sum_{l=0}^{\infty}(2l+1)A_s^I(s)\left[P_l\left(1+\frac{2t}{s-4\mu_0^2}\right)+(-)^IP_l\left(1+\frac{2u}{s-4\mu_0^2}\right)\right]$$
$$\equiv\tilde{A}^I(s,t)+(-)^I\tilde{A}^I(s,u)\,. \tag{8}$$

The requirement of Bose statistics is automatically satisfied; namely, $A_s^I(s,t,u)=(-)^IA_s^I(s,u,t)$. Also, as continuation in l is made, proper "signature" is provided for.

The partial-wave amplitude is defined by

$$A_s^{I,l}(s)=\frac{S^I(l,s)-1}{2ip(s)}\,, \tag{9}$$

[10] T. Regge, Nuovo Cimento 14, 951 (1959).

where the phase-space factor

$$\rho(s)=\left(\frac{s-4\mu_0^2}{s}\right)^{1/2}$$

factors out all the kinematical singularities from the amplitude; the Watson-Sommerfeld transforms are made on the $\tilde{A}$ amplitudes. As usual we obtain

$$\tilde{A}^I(s,t(u))=\frac{1}{\rho(s)}\times\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} dl(2l+1)\frac{S^I(l)-1}{4}\frac{P_l\left(-1-\dfrac{2t(u)}{s-4\mu_0^2}\right)}{\sin\pi l} -\pi\sum_{\mathrm{Re}\alpha_n^I>-\frac{1}{2}}(2\alpha_n^I+1)\beta_n^I\frac{P_{\alpha_n^I}\left(-1-\dfrac{t(u)}{s-4\mu_0^2}\right)}{\sin\pi\alpha_n^I}. \quad (10)$$

We now would like to obtain a relativistic analog of the modified Cheng representation; namely, an expression for the partial-wave S matrix in terms of the relativistic input trajectories, with the appropriate "modification." In writing any representation for the partial-wave amplitude, one has to make assumptions, implicitly perhaps, about its asymptotic behavior in the l plane, or equivalently, about the asymptotic behavior of all its Regge trajectories, even though one may be interested in computing or bootstrapping only the top trajectory on which physical bound states or resonances may lie.[11]

For a superposition of complex, energy-dependent Yukawa potentials restricted by $|V(r,s)|<Mr^{-2}$ we can write[12,13]

$$\alpha_n(s)\underset{s\to\infty}{\longrightarrow}-n+c+\frac{ih^2}{2s^p}P_{n-1}(\cosh\xi), \quad (11)$$

where $n=1, 2, \cdots$, p is a real positive number and $\cosh\xi=1+8\mu_0^2/(s-4\mu_0^2)$. The asymptotic form (11) is somewhat restrictive. Since the asymptotic behavior of the trajectories, together with their assumed analytic properties, may completely determine the trajectories,[8] one may in principle consider the subtraction parameters, obtained from known indeterminacy points in Ref. 8, and c, p, and μ_0 as the only input.

For a much larger class of Yukawa potentials, however, one obtains

$$\alpha_n(s)\underset{s\to\infty}{\longrightarrow}-n+c+\frac{ih^2}{2s^p}\int_{\mu_0}^{\infty}\sigma(\mu')P_{n-1}(1+8\mu'^2/(s-4\mu'^2))d\mu'. \quad (11')$$

This yields an extension of the representation, given in Appendix C. A suitable parametrization of $\sigma(\mu)$ then allows for considerably more input.

In what follows, the exposition is in terms of the more restrictive form (11), which is sufficient to demonstrate many qualitative features of the proposal; in realistic calculations, a more general ansatz such as (11′) is undoubtedly necessary.

The parameters p and c of (11) can be simply related to the behavior of the potential near $r=0$. Furthermore, for potentials which have the behavior r^{-1-q} near $r=0$, the end points of the trajectories are equally spaced.[14] For example, when $q=0$ (a Yukawa like potential), $c=0$, and if there is no energy dependence, $p=\frac{1}{2}$. Actually, in what follows, we need a somewhat weaker assumption that (11); namely, that c may even be a function of n but such that for large n the spacings again become equal. This is satisfied, for example, by the Klein-Gordon and Dirac trajectories with an r^{-1} type of interaction.[15] We hasten to add that through (11) we are not implying that the trajectory has necessarily no left-hand cut, as a relativistic trajectory, even the top one, may. Thus, the end points of $\alpha_n(s)$ for $s\to-\infty$ may well be quite different.

Assuming then that the input trajectories have the asymptotic behavior (11), we have from the Cheng representation

$$\ln S(l,s)\underset{s\to\infty}{\longrightarrow}\sum\frac{\exp[(c-n-l)\xi]}{c-n-l}(\alpha_n^*-\alpha_n) =\frac{ih^2}{s^p}Q_{l-c}(\cosh\xi). \quad (12)$$

Using (12), we follow our derivation of the nonrelativistic modified Cheng representation, and deduce in Appendix B the following representation for the

[11] Self-consistent calculations of the top trajectory within the framework of potential theory are given in D. Hankins, P. Kaus, and C. J. Pearson, Phys. Rev. **137**, B1034 (1965). Similar calculations including both the first and second trajectories have been performed by C. J. Pearson, Ph.D. thesis, University of California, Riverside, 1965 (unpublished). The results of the latter indicate that it is a nontrivial matter to include more than one trajectory in calculations of this type.

[12] J. M. Cornwall and M. A. Ruderman, Phys. Rev. **128**, 1474 (1962).

[13] This general form of a trajectory is shared by relativistic trajectories as shown by V. N. Gribov and I. Y. Pomeranchuk, Zh. Eksperim. i Teor. Fiz. **43**, 308 (1962) [English transl.: Soviet Phys.—JETP **16**, 220 (1963)], under very general assumptions with no regard to an underlying Hamiltonian.

[14] H. Bethe and T. Kinoshita, Phys. Rev. **128**, 1418 (1962), and R. Newton, J. Math. Phys. **3**, 867 (1962).

[15] G. S. Guralnik and C. R. Hagen, Phys. Rev. **130**, 1259 (1963).

partial-wave S matrix:

$$\ln S(l,s)=\sum_n \left\{\int_{\alpha_n}^{\alpha_n^*} \frac{\exp[(l'-l)\xi]}{l'-l}dl' -\frac{ih^2}{(s-4\mu_0^2)^p}\frac{\exp[-(l+c+n)\xi]}{l-c+n}P_{n-1}(\cosh\xi)\right\} +\frac{ih^2}{(s-4\mu_0^2)^p}Q_{l-c}(\cosh\xi)\,, \quad (13)$$

where

$$\cosh\xi=1+\frac{32\gamma\mu_0^2}{s-4\mu_0^2}.$$

As pointed out in Appendix B, γ is a parameter to be determined self-consistently. In Appendix B, we also verify that in (13) we have the proper "modification." That is to say, (13) reduces to (12) as $s\rightarrow\infty$ and also that near threshold we still have the correct behavior for both the real and imaginary parts of the S matrix.

For physical l, using the usual partial-wave projection

$$A_l(s)=\frac{1}{2}\int_{-1}^{1} d(\cos\theta)P_l(\cos\theta)A(s,\cos\theta) \quad (14)$$

and the crossing relations

$$A^I(s,t,u)=\sum_{I'=0}^{2}\chi_{II'}A^{I'}(t,s,u) =(-)^I\sum_{I'=0}^{2}\chi_{II'}A^{I'}(u,s,t)\,, \quad (15)$$

where

$$\chi_{II'}=\begin{bmatrix} \frac{1}{3} & 1 & 5/3 \\ \frac{1}{3} & \frac{1}{2} & -\frac{5}{6} \\ \frac{1}{3} & -\frac{1}{2} & \frac{1}{6}\end{bmatrix}, \quad (16)$$

it is straightforward to obtain for $s<0$

$$\mathrm{Im}A_l^I(s)=\frac{1}{2q_s^2}\int_{4\mu_0^2}^{-4q_s^2} dt\, P_l\left(1+\frac{t}{2q_s^2}\right) \times\sum_{I'=0}^{2}\chi_{II'}\,\mathrm{Im}A^{I'}(t,\cos\theta_t)\,, \quad (17)$$

where

$$\cos\theta_t=1+2s/(t-4\mu_0^2).$$

Since the integral in (17) runs only over values of $t>4\mu_0^2$ ($q_s^2<0$), we only need $\mathrm{Im}A^I(s,z_s)$ for physical s and all z_s. Given the input trajectory then, this can be evaluated using (10) and (11). Thus, essentially the "Born term" is obtained:

$$B_l^I(s)=\frac{1}{\pi}\int_{-4\mu_0^2}^{-\infty}\frac{\mathrm{Im}A_l^I(s')}{s'-s}ds'. \quad (18)$$

Now we can set up the N/D equations to solve for the "output" partial-wave amplitude $A_l(s)$ for all s. This may then be compared with the "input" $A_l(s)$ obtained through the trajectory. The self-consistency problem then simply reduces to varying the trajectory parameters until the two amplitudes match. This argument may easily be extended to the whole l plane by replacing the partial-wave projections in (14) and (17) by the so-called Wong continuation[16]; the Froissart continuation is valid only in the region of the l plane where $\mathrm{Re}l>\mathrm{Max\ Re}\alpha$.

IV. AN ALTERNATIVE METHOD FOR IMPOSING CROSSING

In this section we outline an alternative procedure for approximately satisfying the crossing relations. This differs from that discussed in the previous section in the sense that rather than calculating through crossing the output trajectory resulting from a given trajectory or "potential," and then comparing, here we rather work with the trajectories directly and see to what extent they cause the crossing relations to be satisfied. That is, (13) may be used via the Watson-Sommerfeld transformation to obtain a total amplitude valid for $s>4\mu_0^2$ and all t; in the crossed reaction it will be valid for $t>4\mu_0^2$ and all s.[17] Consequently, there is a common region of validity and one may attempt to adjust all the trajectory parameters until both sides of the crossing relations are satisfied for some region of the (s,t) plane. One might do this in a way analogous to Ref. 18, in which a certain "figure of merit" is achieved by minimizing some function of the number of points at which crossing is to be imposed.

Apart from being operationally much simpler than earlier ones, this method has two other virtues to recommend itself. Until $s=16\mu_0^2$, elastic unitarity is respected, so presumably there is a region (unphysical) of the s-t plane where our expressions, derived on the basis of elastic unitarity alone, should work. Therefore, so long as we are in the region $4\mu_0^2<s,t<16\mu_0^2$, we do not have to incorporate inelasticity. Secondly, we do not have to make any statements, or approximations, about the $I=2$ channel since a crossing condition can be written simply in terms of the $I=0$ and $I=1$ channels alone.

The only input here are the trajectories themselves, and with the exception of the pion mass, it is assumed that all parameters must be determined by some approximate satisfaction of the crossing relations, if indeed they are even sufficient for such determination. For $p>0$ the theory is fully convergent and no arbitrary parameters seem to be necessary. We have tested the

[16] G. F. Chew, Phys. Rev. **129**, 2363 (1963).

[17] The possibility of analytically continuing the modified Cheng representation to negative values of the energy is under investigation.

[18] R. E. Kreps, J. F. Cook, J. J. Brehm, and R. Blankenbecler, Phys. Rev. **133**, B1526 (1964).

case $p=\frac{1}{2}$, $c=0$ of (13), in a one trajectory approximation, and find good agreement with exact results for the partial waves[6] and for the total amplitude, the latter results being discussed in the next section.

Having verified that (13) gives good results for the case $p=\frac{1}{2}$ and $c=0$ in a situation where the answer is known; namely, in potential theory, one may feel a little less hesitant about proceeding to unphysical values of s and t which satisfaction of the crossing relations would require, and moreover, a region in which there are no exact results to compare with. Optimistically, one might hope that the use of whole trajectories, rather than a single point on a trajectory, might make such satisfaction possible, and consequently allow one to make more positive statements with respect to the bootstrap hypothesis and indeed to whether satisfaction of the crossing relations leads to the same world chosen by nature.

One region of the (s,t) plane is eminently suitable for a direct test of our ansatz regarding the trajectories, pending a complete solution of our bootstrap equations. What we are alluding to of course is where one variable, say t, is large and the other, say s, is constrained to be very small.

Let us therefore consider (10) for t large and s small. Then, as is well known, the leading Regge pole term dominates and we have

$$A_R{}^I(s,t)=-\pi[2\alpha_{(s)}{}^I+1]\beta^I(s)\times\frac{P_{\alpha^I(s)}(-1-2t/(s-4\mu_0{}^2))}{\sin\pi\alpha^I(s)}. \quad (19)$$

Also, in the t channel, for large t, we have the analog of (12) for the partial-wave amplitude with our assumed asymptotic behavior of the trajectories. This partial-wave amplitude can be put into the background integral in (10), written appropriately for the t channel, remembering that there is no Regge term for large enough t. Specifically,

$$A_B{}^I(t,s)=\frac{1}{\rho(t)}\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty}dl\left(\frac{2l+1}{4}\right)\times\left\{\exp\left(\frac{ih^2}{t^p}Q_{l-c}(\cosh\xi_t)\right)-1\right\}\times\frac{P_l(-1-2s/(t-4\mu_0{}^2))}{\sin\pi l}. \quad (20)$$

An approximate imposition of crossing can then be implemented between (19) and (20) to determine the trajectory parameters. To be able to see more clearly what is going on let us restrict ourselves to the special case of the parameter $c=0$; this means that the trajectories actually end up at negative integers. In this case, (20) can be solved in closed form to obtain for small s and large t

$$\tilde{A}_B(t,s)=\frac{h^2}{4}\frac{t^{1-p}}{(4\mu_0{}^2-s)}. \quad (21)$$

Various qualitative features may now be verified; in the limit of large t, (19) contains a behavior of the form $t^{\alpha(s)}$. Furthermore, according to the Froissart limit we have $\alpha(0)\leqslant 1$.[19] Therefore, in the region of small s, where (21) is valid, we see that (19) and (21) are consistent for $p\geqslant 0$. In the same spirit, if we now demand that the threshold behavior be correct, as is done in Appendix B, we get $p\lesssim\frac{1}{2}$, since $c=0$, which seems to imply an $\alpha(0)\gtrsim\frac{1}{2}$; specializing to the $I=1$ case, this is a number consistent with various phenomenological estimates for the ρ trajectory. For the $I=0$ case, this would imply that either $p=0$ or the Pomeranchuk trajectory does not end up at -1.

That our scheme has some germs of truth and hence may have a chance of succeeding; i.e., an approximate satisfaction of crossing symmetry may indeed be possible, can also be seen by explicitly using crossing to obtain the contribution to the "Born term," $B_l(t)$ for large t, from $A(s,t)$ for large t and small s, and then obtain an $A(t,s)$. It is of the same general form as (21).

To see this, let us write down $B_l(t)$ for large t, suppressing isospin symbols for the moment, in terms of $A(s,t)$ using crossing:

$$B_l(t)=\frac{1}{\pi}\frac{\chi}{t-4\mu_0{}^2}\int_{-\infty}^{0}ds\,A(s,t)\,\mathrm{Im}Q_l\left(1+\frac{2s}{t-4\mu_0{}^2}\right). \quad (22)$$

For large t, we can use (19) for $A(s,t)$ and because of the $(-t)^{\alpha(s)}$ factor, the leading term in $B_l(t)$ is obtained from small s contributions. Hence approximately, we can write

$$B_l(t)\underset{t\text{ large}}{\approx}t^{\alpha(0)-1}f(t)Q_l\left(1+\frac{8\mu_0{}^2}{t-4\mu_0{}^2}\right), \quad (23)$$

where $f(t)$ presumably has a very weak t dependence. Equation (23) resembles our "input" partial-wave amplitude for large t, as can be seen from (12) with $c=0$. As above, this $B_l(t)$ would then obtain an $A(t,s)$ of the form given in (21). This argument therefore furthers our belief that so long as our "modification" is determined by the input trajectories themselves, our system of equations is internally consistent, at least approximately.

We may remark that a form similar to (23) was written down on the basis of reasonableness by Bander and Shaw[20] in their calculation of the ρ-trajectory parameters.

[19] M. Froissart, Phys. Rev. **123**, 1053 (1961).

[20] M. Bander and G. Shaw, Phys. Rev. **135**, B267 (1964). Recently, J. D. Amand and G. C. Joshi, Nuovo Cimento **38**, 1588 (1965) have shown by a suitable modification of the N/D procedure, that it is possible to obtain a self-consistent solution for the ρ parameters; $m_\rho\approx 400$ MeV and $\Gamma_\rho\approx 167$ MeV. Bander and Shaw were unable to obtain a solution.

TABLE I. In this table are presented the numerical results which have been plotted in Figs. 1–6 as well as an additional column which gives the first term of (5) of the text (background integral). The column labeled "Regge term" was calculated from the second term of (5). The column labeled "calculated amplitude" is the sum of the preceding two; i.e., Eq. (5) with one modified Cheng trajectory as input, the exact trajectory being used. The last column is the exact amplitude as explained in the text.

		Background integral		Regge term		Calculated amplitude		Exact amplitude	
s	z	Re	Im	Re	Im	Re	Im	Re	Im
5.0	0.9	0.833	0.291	0.0	0.0	0.833	0.291	0.830	0.291
	0.6	0.293	0.208	0.0	0.0	0.293	0.208	0.295	0.208
	0.3	0.161	0.164	0.0	0.0	0.161	0.164	0.162	0.164
	0.01	0.104	0.137	0.0	0.0	0.104	0.137	0.103	0.137
	−0.3	0.071	0.116	0.0	0.0	0.071	0.116	0.072	0.116
	−0.6	0.051	0.101	0.0	0.0	0.051	0.101	0.048	0.101
	−0.9	0.037	0.090	0.0	0.0	0.037	0.090	0.036	0.090
1.0	0.9	−0.611	−1.144	1.927	1.920	1.316	0.776	1.311	0.780
	0.6	−0.202	−0.639	0.987	1.365	0.785	0.726	0.785	0.726
	0.3	−0.149	−0.464	0.663	1.144	0.514	0.680	0.514	0.680
	0.01	−0.132	−0.368	0.484	1.009	0.352	0.641	0.352	0.641
	−0.3	−0.122	−0.302	0.356	0.905	0.234	0.603	0.234	0.603
	−0.6	−0.115	−0.256	0.267	0.827	0.152	0.571	0.151	0.571
	−0.9	−0.110	−0.221	0.198	0.763	0.880	0 542	0.088	0.542
0.1	0.9	−1.357	0.377	2.025	2.743	0.688	3.120	0.676	3.121
	0.6	−0.621	0.182	1.187	2.936	0.566	3.118	0.566	3.118
	0.3	−0.379	0.117	0.847	2.998	0.467	3.115	0.468	3.115
	0.01	−0.254	0.081	0.635	3.032	0.381	3.113	0.381	3.113
	−0.3	−0.172	0.057	0.468	3.053	0.296	3.110	0.296	3.110
	−0.6	−0.119	0.040	0.340	3.067	0.221	3.107	0.221	3.108
	−0.9	−0.083	0.028	0.234	3.077	0.151	3.105	0.151	3.105

V. DISCUSSION OF NUMERICAL RESULTS AND CALCULATIONS

In this section we discuss the numerical results presented in Table I and in Figs. 1–6. The basic equation being tested is (13) of the text. It is being studied with the following specific values of the parameters, $p=\frac{1}{2}$, $h^2=1.8$, $c=0$, $\gamma=1$, and nonrelativistic kinematics. For this situation the representation may be compared with potential theory where exact results are known[21] for the particular potential $V(r)=-1.8e^{-r}/r$. In a previous paper[6] results for (13) were presented for these values of the parameters and compared with the exact results. In this paper we wish to further test the representation (13) by using it to calculate the total scattering amplitude $f(s,z)$ for $z=\cos\theta$ in the physical region. This may be accomplished by inserting the modified Cheng representation for the partial-wave S matrix $S(l,s)$ given by (13) into the background integral of (5) and evaluating $f(s,z)$ for various (s,z) pairs. The only input here is the exact top trajectory $\alpha(s)$ as given in Ref. 21. Once the trajectory has been specified, the residue $\beta(s)$ is determined and may be calculated directly from (13).

The background integral in (5) was evaluated by first transforming the region $(-\frac{1}{2}-i\infty, -\frac{1}{2}+i\infty)$ into

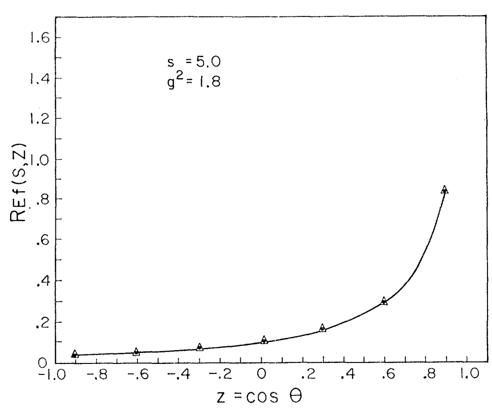

FIG. 1. The real part of the total scattering amplitude for an attractive Yukawa potential of strength $g^2=1.8$, unit range, and energy $s=5$ is plotted as a function of $z=\cos\theta$ where θ is the scattering angle. The exact calculation and the background integral plus Regge term [Eq. (5) of the text] coincide to within plotting accuracy.

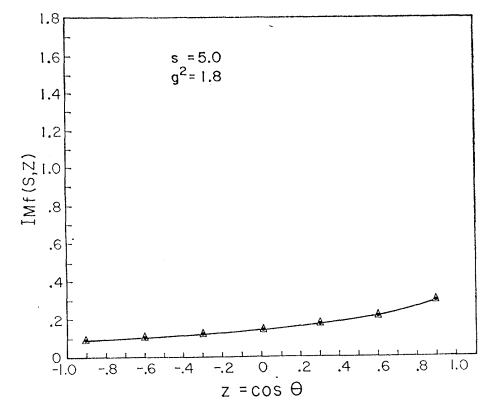

FIG. 2. Same as Fig. 1 except the imaginary part of the amplitude is plotted.

[21] A. Ahmadzadeh, P. Burke, and C. Tate, Phys. Rev. **131**, 1315 (1963).

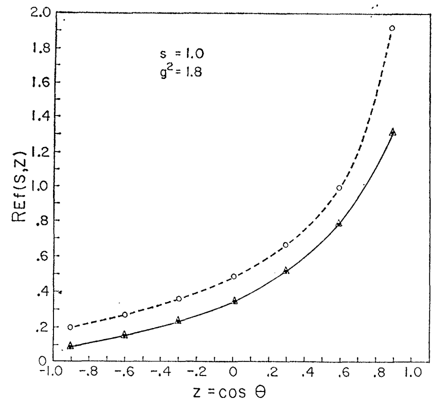

FIG. 3. Triangles and solid curve: same as Fig. 1 for the real part of the amplitude except $s=1$. Circles and dashed curve: calculated from the second term of Eq. (5) of the text (Regge term).

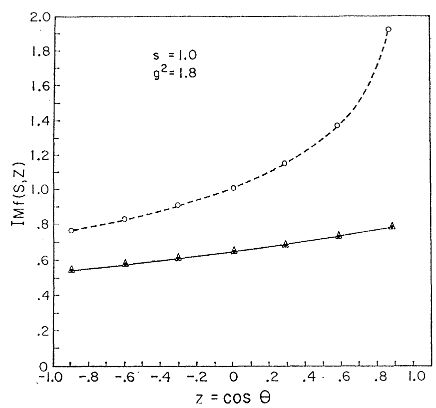

FIG. 4. Same as Fig. 3 except the imaginary part of the amplitude is plotted.

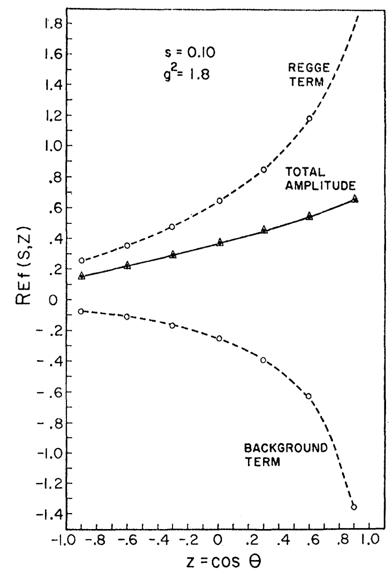

FIG. 5. Same as Fig. 3 except $s=0.1$ and the background term is also plotted.

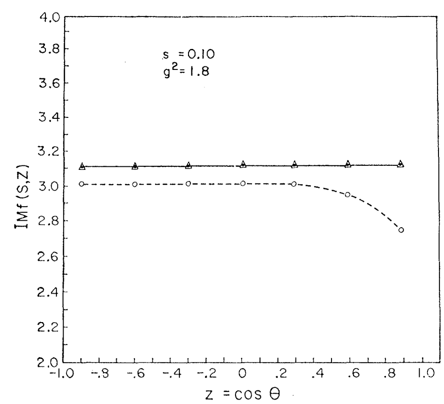

FIG. 6. Same as Fig. 4 except $s=0.1$.

$(-1, 1)$ and then integrating via Simpson's rule. A mesh size of 400 points was used. The integral was tested for stability and it was found that increasing the mesh size from 400 to 600 points caused a change in the third or fourth decimal place. The integrand oscillates rapidly near the end points of integration, and while the factor $P_l(-z)/\sin\pi l$ provides exponential damping in the physical region of z, high accuracy is not easy to achieve. On the other hand, the various other functions required; namely, $P_l(-z)$, $Q_l(z)$, and $\mathrm{Ei}(z)$ were tested in various ways such as satisfaction of recurrence relations and evaluation at special points, and it was found that they can be calculated via their respective integral representations to about 6–7 place accuracy with 16-point Gaussian formulas. Before discussing the actual numerical results, we may add that it is not feasible to use (5) to calculate $f(s,z)$ for z on the cut $1\leq z<+\infty$ as the quantity $P_l(-z)/\sin\pi l$ will no longer provide an exponential damping but undergoes pure oscillations. This difficulty may be overcome by explicitly exhibiting the singularities of $f(s,z)$ in the z plane as is done in Appendix A.

The actual numerical results are presented in Table I and graphically in Figs. 1–6. The column labeled "background integral" of Table I gives the values of the first term of (5). The one labeled "Regge term" is the second term of (5). These have been plotted as dashed curves on the graphs, Figs. 3–6 where the energy is such that $\mathrm{Re}\alpha>-\frac{1}{2}$; namely, for $s=1$ and $s=0.1$. At the other energy studied $s=5$, $\mathrm{Re}\alpha\approx-0.97$ and the single Regge term no longer contributes to $f(s,z)$.

The column labeled "calculated amplitude" contains the results for Eq. (5) of the text. The real and imaginary parts of $f(s,z)$ are plotted versus $z=\cos\theta$ for $s=5$ as shown in Figs. 1 and 2, respectively; as stated above, no Regge term contributes here and consequently none is shown. In other words, all of the amplitude comes from the background integral in this case. In Figs. 3 and 4 similar results are shown for $s=1$; Figs. 5 and 6 show the corresponding results for $s=0.1$. The latter two cases; namely, $s=1$ and $s=0.1$ both show the real and imaginary parts of the Regge term and it is clear from the graphs Figs. 3–6 that for these values of s and z the statement that the amplitude may be approximated by a single Regge term, without its contribution to the background integral, is certainly not true. We also see from Figs. 1, 3, and 5 that $\mathrm{Re}f(s,z)$ near $z=1$ is increasing toward its Born pole value, which for larger s values, is just outside the physical region.

The column of Table I labeled "exact amplitude" was obtained by calculating eleven terms of the partial-wave expansion. In this calculation the modified Cheng representation for $S(l,s)$ was used, although only the leading trajectory contributes, the contribution from the second trajectory being in the fourth decimal place; the error due to terminating the partial-wave expansion at eleven terms varies from the fourth to the seventh decimal place. Ahmadzadeh has calculated $S(l,s)$ exactly up to D waves and when a comparison can be made, we are in complete agreement with his results.[22]

VI. CONCLUSION

In conclusion, we have proposed a program for bootstrapping entire Regge trajectories, which is free from any undetermined arbitrary parameters. The final success of any bootstrap program based on Regge trajectories must of necessity be determined by the convergence of the theory in terms of the number of trajectories. With the present computational facilities available, one may attempt a bootstrap with one or perhaps two trajectories, but probably nothing beyond that. In this context, it then becomes imperative to have a faithful amplitude in terms of a few trajectories. We have constructed an approximate amplitude which has built in analytic structure demanded by the Mandelstam representation, is unitary and also retains the correct threshold behavior. The total amplitude numerically computed retaining just one top trajectory gives good agreement compared with the exact in situations where such a comparison is possible; namely, in potential theory. It is our hope that using the above representation we can bootstrap entire Regge trajectories, and this is presently under investigation.

Convergence in terms of the number of trajectories can of course be tested numerically. The parameters of the asymptotic trajectories c, p, and h^2 are determined from consistency. The effect of the particular chosen asymptotic form, however, is a much subtler question. One would require a knowledge of which forms, if any, permit a self-consistent solution and what difference they make. The particular form (11) was chosen, because it applies to a large class of energy-dependent potentials within the framework of two-particle unitarity.

ACKNOWLEDGMENT

The calculations were carried out at the Computing Center, University of California, Riverside, California.

[22] A. Ahmadzadeh, Phys. Rev. **133**, B1074 (1964).

APPENDIX A: PROPERTIES OF THE CONSTRUCTED TOTAL SCATTERING AMPLITUDE (NONRELATIVISTIC) IN THE Z PLANE

Here we make explicit some of the properties of $f(s,z)$ advertised in Sec. II.

For a Yukawa potential, the first Born term

$$A_B(\lambda,s)=(g^2/2s)Q_{\lambda-\frac{1}{2}}(\cosh\xi) \tag{A1}$$

so that upon decomposing the amplitude

$$A(\lambda,s)=[A(\lambda,s)-A_B(\lambda,s)]+A_B(\lambda,s)$$

we obtain for the total amplitude:

$$f(s,z)=\sum_\lambda (2\lambda)[A(\lambda,s)-A_B(\lambda,s)]P_{\lambda-\frac{1}{2}}(z)+\frac{g^2}{2s^{1/2}(z_0-z)}, \tag{A2}$$

where $z_0=1+\mu^2/2s$. The last term is the celebrated "Born pole," $f_B(s,z)$. In what follows, Re and Im refer to the unitarity cut, not the z cut. Now, let us write

$$\mathrm{Im}A(\lambda,s)=\frac{1}{2i}\{A(\lambda,s)-A^*(\lambda^*,s)\}=s^{1/2}A(\lambda,s)A^*(\lambda^*,s)$$

so that if $A(\lambda,s)$ is bounded by $e^{-\lambda\xi}/\sqrt{\lambda}$ for large λ, $\mathrm{Im}A(\lambda,s)$ is bounded by $e^{-2\lambda\xi}/\lambda$. Thus, we take the real and imaginary parts of $f(s,z)$ in (A2) and obtain the Watson-Sommerfeld transform for each of them separately.[5] Thus,

$$f(s,z)=\frac{g^2}{2s^{1/2}(z_0-z)}-i\int_{-i\infty}^{i\infty}\lambda d\lambda\, P_{\lambda-\frac{1}{2}}(-z)\frac{[A(\lambda,s)-A_B(\lambda,s)]}{\cos\pi\lambda}+\pi\sum_n\frac{(2\lambda_n)\beta_n}{\cos\pi\lambda_n}P_{\lambda_n-\frac{1}{2}}(-z). \tag{A3}$$

Inserting the integral representation for $P_\lambda(-z)$,

$$\frac{\pi\lambda P_{\lambda-\frac{1}{2}}(-z)}{\cos\pi\lambda}=\frac{1}{2^{3/2}}\int_{-\infty}^{\infty}\frac{e^{\lambda x}\sinh x}{(\cosh x-z)^{3/2}}dx, \tag{A4}$$

we obtain

$$\mathrm{Re}f(s,z)=\frac{g^2}{2s^{1/2}(z_0-z)}+\sqrt{2}\int_\xi^\infty\frac{B_R'(x)}{(\cosh x-z)^{1/2}}dx+\mathrm{Re}\sum_n\left\{(2\alpha_n+1)\beta_n\left[\frac{\pi P_{\alpha_n}(-z)}{\sin\pi\alpha_n}+\frac{1}{\sqrt{2}}\int_{-\infty}^{\xi}\frac{e^{\lambda_n x}}{(\cosh x-z)^{1/2}}dx\right]\right\}, \tag{A5}$$

where

$$B_R(x,s)=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}e^{\lambda x}\left\{\frac{A(\lambda,s)+A^*(\lambda^*,s)}{2}-A_B(\lambda,s)\right\}d\lambda. \quad (A6)$$

Similarly, we obtain for $\mathrm{Im}f(s,z)$:

$$\mathrm{Im}f(s,z)=\sqrt{2}\int_{2\xi}^{\infty}\frac{B_I'(x)}{(\cosh x-z)^{1/2}}dx+\mathrm{Im}\sum_n\left\{(2\alpha_n+1)\beta_n\left[\frac{\pi P_{\alpha_n}(-z)}{\sin\pi\alpha_n}+\frac{1}{\sqrt{2}}\int_{-\infty}^{2\xi}\frac{e^{\lambda_n x}}{(\cosh x-z)^{1/2}}dx\right]\right\}, \quad (A7)$$

where

$$B_I(x,s)=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}e^{\lambda x}\left\{\frac{A(\lambda,s)-A^*(\lambda^*,s)}{2i}\right\}d\lambda. \quad (A8)$$

Since

$$\cosh\xi=1+\frac{\mu^2}{2s},$$

we have

$$\cosh 2\xi=1+\frac{2\mu^2}{s}+\frac{\mu^4}{2s^2}\equiv 1+\frac{t_0}{2s},$$

where

$$t_0=4\mu^2+\mu^4/s.$$

Thus, the representation for $f(s,z)$ through (A5) and (A7) has the following analytic structure:

(1) It has the Born pole at $z=z_0$.

(2) It has a cut in the z plane starting at $z=1+2\mu^2/s$.

(3) The spectral function $\rho(s,t)$, i.e., the discontinuity in t of the discontinuity in s of the function $f(s,t)$ is nonzero in a region bounded by the correct curved boundary given by $t_0=4\mu^2+\mu^4/s$.

APPENDIX B: DERIVATION OF A RELATIVISTIC ANALOG OF THE MODIFIED CHENG REPRESENTATION AND ITS ASYMPTOTIC AND THRESHOLD PROPERTIES

Given (12) of the text, we follow our derivation of the nonrelativistic modified Cheng representation[6] and consider the following integral:

$$I=\frac{1}{2\pi i}\int_{C'}\frac{dl'}{l'-l}(\exp l'\xi)\left[\ln S(l',s)-\frac{ih^2}{(s-4\mu_0^2)^p}Q_{l'-c}(\cosh\xi)\right], \quad (B1)$$

where C' is an infinite circle in the l plane. Let us assume that a parameter γ exists, such that for $\mathrm{Re}l'>-\frac{1}{2}$, the difference in brackets in (B1) is bounded by $[\exp(-l'\xi)]/l'^{\epsilon}$, as $|l'|\to\infty$ ($\epsilon>0$) where $\cosh\xi=1+32\gamma\mu_0^2/(s-4\mu_0^2)$ and γ is as yet undetermined. For the class of potentials for which the modified Cheng representation was proved, $\ln S(l',s)$ was bounded by $2\pi il'$ when $|l'|\to\infty$ with $\mathrm{Re}l'<-\frac{1}{2}$. Even if the actual bound were much worse, the integrand goes to zero as $|l'|\to\infty$ with $\mathrm{Re}l'<-\frac{1}{2}$. Thus, $I=0$ and as before we obtain:

$$\ln S(l,s)=\sum_n\left\{\int_{\alpha_n}^{\alpha_n^*}\frac{\exp[(l'-l)\xi]}{l'-l}dl'-\frac{ih^2}{(s-4\mu_0^2)^p}\frac{\exp[-(l-c+n)\xi]}{l-c+n}P_{n-1}(\cosh\xi)\right\}+\frac{ih^2}{(s-4\mu_0^2)^p}Q_{l-c}(\cosh\xi). \quad (B2)$$

For large s, with the assumed behavior of the input trajectories as in (11) we have

$$\int_{\alpha_n}^{\alpha_n^*}\frac{\exp[(l'-l)\xi]}{l'-l}dl'-\frac{ih^2}{(s-4\mu_0^2)^p}\frac{\exp[-(l-c+n)\xi]}{l-c+n}\times P_{n-1}(\cosh\xi)\underset{s\to\infty}{\longrightarrow}0$$

and consequently

$$\ln S(l,s)\underset{s\to\infty}{\longrightarrow}\frac{ih^2}{s^p}Q_{l-c}(\cosh\xi)\underset{s\to\infty}{\longrightarrow}0\quad\text{for}\quad p>0$$

as it should.

For small s, we have already shown in Ref. 6 that for trajectories with

$$\alpha_n\underset{s\to4\mu_0^2}{\longrightarrow}\alpha_0+O[(s-4\mu_0^2)^{\alpha_0+\frac{1}{2}}],$$

the integral term in (B2) has the proper $(s-4\mu_0^2)^{l+\frac{1}{2}}$ behavior. Thus, the proper threshold behavior for the total phase shift would be insured if the other terms in (B2) go like $(s-4\mu_0^2)^{l+\frac{1}{2}}$ or higher. We have

$$(s-4\mu_0^2)^{-p}\exp[-(l-c+n)\xi]\times P_{n-1}(\cosh\xi)\underset{s\to4\mu_0^2}{\longrightarrow}(s-4\mu_0^2)^{l+1-c-p}$$

and the same behavior for the Q term. Therefore, if $p\leqslant\frac{1}{2}-c$, the proper threshold behavior is guaranteed. Furthermore, if $p>0$ then $c<\frac{1}{2}$. We expect $c=\frac{1}{2}$ to be a critical case since this corresponds to a potential of the form r^{-2}. Thus, the proper threshold and asymptotic behavior is verified.

APPENDIX C: DERIVATION OF THE MODIFIED CHENG REPRESENTATION FOR A SUPERPOSITION OF YUKAWA POTENTIALS AND ITS PROPERTIES

In Ref. 6 and in this paper, we have made the assertion that, for a superposition of Yukawa potentials, with exponentially decreasing weight factor, the correct generalization of (1) of the text is the replacement of the Q function by the integral over the weight factor, and that the resulting total amplitude correctly has a cut beginning at the Born-pole position. The purpose of this Appendix is to prove these assertions under the *assumption* that the partial-wave S matrix for this potential obey the result found by Cheng and Wu[23]; namely, that[24]

$$S(\lambda,s) \xrightarrow[|\lambda|\to\infty]{} e^{2\pi i\lambda}, \quad \mathrm{Re}\lambda<0. \tag{C1}$$

Consider a potential of the form

$$V(r)=-g^2\int_\mu^\infty \sigma(\mu')\frac{e^{-\mu' r}}{r}d\mu', \tag{C2}$$

where

$$\sigma(\mu') \xrightarrow[\mu'\to\infty]{} e^{-\epsilon\mu'} \quad \text{for some} \quad \epsilon>0. \tag{C3}$$

Furthermore, consider the integral

$$I=\frac{1}{2\pi i}\int_C \frac{d\lambda' \exp(\lambda'\xi)}{\lambda'-\lambda}\left[\ln S(\lambda',s) -\frac{ig^2}{\sqrt{s}}\int_\mu^R \sigma(\mu')Q_{\lambda'-\frac{1}{2}}(\cosh\xi')d\mu'\right], \tag{C4}$$

where

$$\cosh\xi'=1+\mu'^2/2s, \tag{C5}$$

and

$$\cosh\xi=1+(2\mu)^2/2s \tag{C6}$$

and

$$\mu\le R<2\mu. \tag{C7}$$

If we abbreviate by F

$$F=\ln S(\lambda',s)-\frac{ig^2}{\sqrt{s}}\int_\mu^R \sigma(\mu')Q_{\lambda'-\frac{1}{2}}(\cosh\xi')d\mu' \tag{C8}$$

then for $\mathrm{Re}\lambda'>0$ we have

$$F \xrightarrow[|\lambda'|\to\infty]{} \frac{\exp(-\lambda'\xi)}{(\lambda')^{3/2}} \quad \mathrm{Re}\lambda'>0 \tag{C9}$$

since the lowest mass (longest range) part of the force dominates in the integral of (C8). On the other hand, for $\mathrm{Re}\lambda'<0$, we have in view of C1

$$F \xrightarrow[|\lambda'|\to\infty]{} \frac{e^{-\lambda'\xi(R)}}{(|\lambda'|)^{1/2}}, \quad \mathrm{Re}\lambda'<0 \tag{C10}$$

where

$$\cosh\xi(R)=1+R^2/2s.$$

Therefore,

$$I=0. \tag{C11}$$

Evaluating I via Cauchy's theorem then yields

$$S(l,s)=\exp\left\{\frac{ig^2}{\sqrt{s}}\int_\mu^R \sigma(\mu')Q_l(\cosh\xi')d\mu'\right\} \times\prod_n S_n(l,s), \tag{C12}$$

where

$$S_n(l,s)=\exp\left\{\int_{\alpha_n}^{\alpha_n^*}\frac{\exp[(l'-l)\xi]}{l'-l}dl' -\frac{ig^2}{\sqrt{s}}\frac{\exp[-(l+n)\xi]}{l+n}\int_\mu^R \sigma(\mu')P_{n-1}(\cosh\xi')d\mu'\right\}.$$

Since the result (C12) is an analytic function of R, and since $\sigma(\mu')$ has the behavior (C3), we may relax the restriction (C7) on R and let $R\to\infty$ in (C12).

The second part of the assertion is now easily proved since from (C12)

$$\ln S(l,s) \xrightarrow[s\to\infty]{} \frac{ig^2}{\sqrt{s}}\int_\mu^\infty \sigma(\mu')Q_l(\cosh\xi')d\mu' \tag{C13}$$

and

$$f_{\mathrm{Born}}(s,z)=\sum_{\rho=0}^\infty (2l+1)\frac{ig^2}{\sqrt{s}}\int_\mu^\infty \sigma(\mu')Q_l(\cosh\xi')d\mu' P_l(z) =\frac{ig^2}{\sqrt{s}}\int_\mu^\infty \frac{\sigma(\mu')d\mu'}{(1+\mu'^2/2s)-z}, \tag{C14}$$

which has the correct cut beginning at $z=1+\mu^2/2s$.

[23] H. Cheng and T. T. Wu, Harvard University (to be published).

[24] This result has recently been proved for a very general class of potentials; W. J. Abbe and Y. N. Srivastava (to be published).